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Hybrid Noor iteration method for nonexpansive mappings in Hilbert spaces☆

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ABSTRACT

In this paper, hybrid Noor iteration method was introduced for nonexpansive mappings in Hilbert spaces. The sufficient and necessary conditions that the iteration sequences converge strongly to a fixed point of a nonexpansive mapping are obtained in Hilbert spaces.

1. Introduction and preliminaries

Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. A mapping $T : H \to H$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for any $x, y \in H$. A mapping $F : H \to H$ is said to be η-strongly monotone if there exists a constant $\eta > 0$ such that $\langle Fx - Fy, x - y\rangle \ge \eta\|x - y\|^2$ for any $x, y \in H$. $F : H \to H$ is said to be *k-Lipschitzian* if there exists a constant $k > 0$ such that $\|Fx - Fy\| \le k\|x - y\|$ for any $x, y \in H$.

The interest and importance of construction of fixed points of nonexpansive mappings stem mainly from the fact that it may be applied in many areas, such as imagine recovery and signal processing (see, e.g., [1–3]). Iterative techniques for approximating fixed points of nonexpansive mappings have been studied by various authors (see, e.g., [1–12], etc.), using famous Mann iteration method, Ishikawa iteration method and many other iteration methods. Especially, it is proved that Mann iteration sequence just converges weakly to a fixed point of a nonexpansive mapping, even in Hilbert space.

In 2000, Noor [13] introduced the following three-step iterative scheme to study the approximate solutions for general variational inequalities in Hilbert spaces. Noor iteration is defined as follows:

$$\begin{cases} z_n = a_n x_n + (1 - a_n)Tx_n \\ y_n = b_n x_n + (1 - b_n)Tz_n \quad n \ge 0 \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n \end{cases} \tag{1.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are appropriate sequences in [0,1].

Since then, Noor iteration scheme has been applied to study the strong and weak convergence of nonexpansive mappings and asymptotically nonexpansive mappings (see, e.g., [12,14,15], etc).

Recently, for studying the strong and weak convergence of fixed points of nonexpansive mappings, Wang [7] introduced the following hybrid iteration scheme: For $x_0 \in H$ is given arbitrarily,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^{\lambda_{n+1}} x_n \quad n \ge 0, \tag{1.2}$$

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where $T^{\lambda}x = Tx - \lambda\mu F(Tx)$ for all $x \in H$, λ, μ are two constants and $F : H \to H$ is an η-strongly monotone and *k-Lipschitzian*.

Motivated by those work of Wang and Noor, in this paper we propose the following hybrid Noor iteration scheme:

$$\begin{cases} z_n = a_n x_n + (1 - a_n) T^{\lambda_{n+1}} x_n \\ y_n = b_n x_n + (1 - b_n) T^{\lambda_{n+1}} z_n \quad n \geq 0 \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^{\lambda_{n+1}} y_n \end{cases} \tag{1.3}$$

where $T^{\lambda_{n+1}} x_n = Tx_n - \lambda_{n+1}\mu F(Tx_n)$, $\{\lambda_n\} \subset [0, 1)$, $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\} \subset (0, 1)$, $x_0 \in H$ is chosen arbitrarily.

If $\lambda_n = 0$ for all positive integer n, then (1.3) reduces to Noor iteration (1.1).

If $a_n = 1$ and $\lambda_n = 0$ for all positive integer n, then (1.3) reduces to Ishikawa iteration

$$\begin{cases} y_n = b_n x_n + (1 - b_n) T x_n \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n \end{cases} \tag{1.4}$$

where $\{b_n\}$, $\{\alpha_n\} \subset (0, 1)$.

If $a_n = b_n = 1$ and $\lambda_n = 0$ for all positive integer n, then (1.3) reduces to Mann iteration

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n \tag{1.5}$$

where $\{\alpha_n\} \subset (0, 1)$.

The purpose of this paper is to study the strong and weak convergence theorems of the hybrid Noor iteration scheme to a fixed point of nonexpansive mappings in Hilbert space.

Now, we recall the well-known concepts and results.

A Banach space E is said to satisfy *Opial's* condition [11] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$.

A mapping $T : K \to K$ is said to be semicompact if, for any sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0 (n \to \infty)$, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to $x^* \in K$.

A mapping $T : K \to K$ is said to be demiclosed at the origin, if for each sequence $\{x_n\}$ in D, the condition $x_n \to x_0$ weakly and $Tx_n \to 0$ strongly implies $Tx_0 = 0$.

In what follows, the following lemmas are needed to prove our main results.

Lemma 1.1 ([9]). *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be three nonnegative sequences satisfying*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n, \quad \forall n = 1, 2, \ldots.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 1.2 ([10]). *Let $T^{\lambda}x = Tx - \lambda\mu F(Tx)$, where $T : H \to H$ is a nonexpansive mapping and $F : H \to H$ is an η-strongly monotone and k-Lipschitzian mapping. If $0 \leq \lambda < 1$ and $0 < \mu < 2\eta/k^2$, then T^{λ} is a contraction and satisfies*

$$\|T^{\lambda}x - T^{\lambda}y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in H,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$.

Lemma 1.3 ([11]). *Let $p > 1$, $r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda) y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - \omega_p(\lambda) g(\|x - y\|),$$

for all x, y in $B_r = \{x \in X : \|x\| \leq r\}$, $\lambda \in [0, 1]$, where $\omega_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.

Lemma 1.4 ([12]). *Let K be a nonempty closed convex subset of a real Hilbert space H and $T : H \to H$ a nonexpansive mapping. If T has a fixed point, then $I - T$ is demiclosed at zero, where I is the identity mapping of H, that is, whenever $\{x_n\}$ is a sequence in K weakly converging to some $x \in K$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$.*

2. Main results

Lemma 2.1. *Let H be a Hilbert space, $T : H \to H$ a nonexpansive mapping with $F(T) = \{x : Tx = x\} \neq \emptyset$, and $F : H \to H$ an η-strongly monotone and k-Lipschitzian mapping. For any given $x_0 \in H$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are defined as in (1.3) with the following conditions:*

(i) *$\alpha \leq a_n, b_n, \alpha_n \leq \beta$ for some $\alpha, \beta \in (0, 1)$;*
(ii) *$\sum_{n=1}^{\infty} \lambda_n < \infty$;*
(iii) *$0 < \mu < 2\eta/k^2$.*

Then,

(1) $\lim_{n\to\infty} \|x_n - q\|$ *exists for all* $q \in F(T)$;
(2) $\lim_{n\to\infty} \|T^{\lambda_{n+1}}x_n - x_n\| = 0$, $\lim_{n\to\infty} \|T^{\lambda_{n+1}}y_n - y_n\| = 0$, $\lim_{n\to\infty} \|T^{\lambda_{n+1}}z_n - z_n\| = 0$;
(3) $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, $\lim_{n\to\infty} \|Ty_n - y_n\| = 0$, $\lim_{n\to\infty} \|Tz_n - z_n\| = 0$.

Proof. (1) For each $q \in F(T)$, we have

$$\begin{aligned}\|z_n - q\|^2 &= \|a_n x_n + (1-a_n)T^{\lambda_{n+1}}x_n - q\|^2\\ &= \|a_n(x_n - q) + (1-a_n)(T^{\lambda_{n+1}}x_n - q)\|^2\\ &\le a_n\|x_n - q\|^2 + (1-a_n)\|T^{\lambda_{n+1}}x_n - q\|^2 - a_n(1-a_n)\|T^{\lambda_{n+1}}x_n - x_n\|^2. \end{aligned} \tag{2.1}$$

From Lemma 1.2, we know that

$$\begin{aligned}\|T^{\lambda_{n+1}}x_n - q\| &= \|T^{\lambda_{n+1}}x_n - Tq\|\\ &\le \|T^{\lambda_{n+1}}x_n - T^{\lambda_{n+1}}q\| + \|T^{\lambda_{n+1}}q - Tq\|\\ &\le (1-\lambda_{n+1}\tau)\|x_n - q\| + \lambda_{n+1}\mu\|F(q)\|\\ &= (1-\lambda_{n+1}\tau)\|x_n - q\| + \lambda_{n+1}\tau\left\|\frac{\mu}{\tau}F(q)\right\|. \end{aligned} \tag{2.2}$$

Furthermore,

$$\begin{aligned}\|T^{\lambda_{n+1}}x_n - q\|^2 &\le (1-\lambda_{n+1}\tau)\|x_n - q\|^2 + \lambda_{n+1}\tau\left\|\frac{\mu}{\tau}F(q)\right\|^2\\ &= (1-\lambda_{n+1}\tau)\|x_n - q\|^2 + \frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2. \end{aligned} \tag{2.3}$$

Thus,

$$\begin{aligned}\|z_n - q\|^2 &\le a_n\|x_n - q\|^2 + (1-a_n)(1-\lambda_{n+1})\|x_n - q\|^2 + (1-a_n)\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2\\ &\le \|x_n - q\|^2 + \frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2 \end{aligned} \tag{2.4}$$

$$\begin{aligned}\|y_n - q\|^2 &= \|b_n(x_n - q) + (1-b_n)T^{\lambda_{n+1}}z_n - q\|^2\\ &\le b_n\|x_n - q\|^2 + (1-b_n)\|T^{\lambda_{n+1}}z_n - q\|^2 - b_n(1-b_n)\|T^{\lambda_{n+1}}z_n - x_n\|^2. \end{aligned} \tag{2.5}$$

By (2.4), we have

$$\begin{aligned}\|T^{\lambda_{n+1}}z_n - q\|^2 &\le (1-\lambda_{n+1}\tau)\|z_n - q\|^2 + \frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2\\ &\le (1-\lambda_{n+1}\tau)\left(\|x_n - q\|^2 + \frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2\right) + \frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2\\ &\le (1-\lambda_{n+1})\|x_n - q\|^2 + 2\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2. \end{aligned} \tag{2.6}$$

Therefore,

$$\begin{aligned}\|y_n - q\|^2 &\le b_n\|x_n - q\|^2 + (1-b_n)\left((1-\lambda_{n+1}\tau)\|x_n - q\|^2 + 2\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2\right)\\ &\le \|x_n - q\|^2 + 2\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2 \end{aligned} \tag{2.7}$$

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|\alpha_n(x_n - q) + (1-\alpha_n)(T^{\lambda_{n+1}}y_n - q)\|^2\\ &\le \alpha_n\|x_n - q\|^2 + (1-\alpha_n)\|T^{\lambda_{n+1}}y_n - q\|^2 - \alpha_n(1-\alpha_n)\|T^{\lambda_{n+1}}y_n - x_n\|^2. \end{aligned}$$

From (2.7), we know that

$$\begin{aligned}\|T^{\lambda_{n+1}}y_n - q\|^2 &\le (1-\lambda_{n+1}\tau)\|y_n - q\|^2 + \frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2\\ &\le (1-\lambda_{n+1}\tau)\|x_n - q\|^2 + 3\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2. \end{aligned} \tag{2.8}$$

So,

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \alpha_n\|x_n-q\|^2+(1-\alpha_n)((1-\lambda_{n+1}\tau)\|x_n-q\|^2\\&\quad+3\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2)-\alpha_n(1-\alpha_n)\|T^{\lambda_{n+1}}y_n-x_n\|^2\\&\le \|x_n-q\|^2+3\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2-\alpha_n(1-\alpha_n)\|T^{\lambda_{n+1}}y_n-x_n\|^2.\end{aligned}\tag{2.9}$$

Since $\sum_{n=1}^{\infty}\lambda_n<\infty$, it follows from Lemma 1.1 that $\lim_{n\to\infty}\|x_n-q\|$ exists for each $q\in F(T)$. We also know that $\{x_n\}$ is bounded.

(2) From (2.9), we have

$$\alpha_n(1-\alpha_n)\|T^{\lambda_{n+1}}y_n-x_n\|^2\le\|x_n-q\|^2-\|x_{n+1}-q\|^2+3\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2\tag{2.10}$$

where $\alpha_n(1-\alpha_n)\ge\alpha(1-\beta)$. So

$$\alpha(1-\beta)\|T^{\lambda_{n+1}}y_n-x_n\|^2\le\|x_n-q\|^2-\|x_{n+1}-q\|^2+3\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2.\tag{2.11}$$

From $\sum_{n=1}^{\infty}\lambda_n<\infty$, we know that $\lambda_n\to0\ (n\to\infty)$, and $3\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2\to0\ (n\to\infty)$. Thus, $\lim_{n\to\infty}\|T^{\lambda_{n+1}}y_n-x_n\|^2=0$.

Since H is a Hilbert space, it follows from the above conclusion (1) and (2.6) that $\{x_n-1\}_{n=1}^{\infty}$ and $\{T^{\lambda_{n+1}}z_n-q\}_{n=1}^{\infty}$ are bounded. Therefore, there exists $r>0$ such that $x_n-q\in B_r$, $T^{\lambda_{n+1}}z_n-q\in B_r$ for all positive integer n. By Lemma 1.3, there is a continuous, strictly increasing, and convex function $g:[0,\infty)\to[0,\infty)$, $g(0)=0$ such that

$$\begin{aligned}\|y_n-q\|^2&=\|b_n(x_n-q)+(1-b_n)T^{\lambda_{n+1}}z_n-q\|^2\\&\le b_n\|x_n-q\|^2+(1-b_n)\|T^{\lambda_{n+1}}z_n-q\|^2-\omega_2(b_n)g(\|T^{\lambda_{n+1}}z_n-x_n\|)\\&\le\|x_n-q\|^2+2\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2-\omega_2(b_n)g(\|T^{\lambda_{n+1}}z_n-x_n\|)\end{aligned}\tag{2.12}$$

where $\omega_2(b_n)=b_n^2(1-b_n)+b_n(1-b_n)^2=b_n(1-b_n)$, and

$$\begin{aligned}\|x_{n+1}-q\|^2&\le\alpha_n\|x_n-q\|^2+(1-\alpha_n)((1-\lambda_{n+1}\tau)\|y_n-q\|^2+\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2)\\&\le\alpha_n\|x_n-q\|^2+(1-\alpha_n)(1-\lambda_{n+1}\tau)\|x_n-q\|^2\\&\quad+2(1-\alpha_n)(1-\lambda_{n+1}\tau)\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2+(1-\alpha_n)\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2\\&\quad-(1-\alpha_n)\omega_2(b_n)g(\|T^{\lambda_{n+1}}z_n-x_n\|)\\&\le\|x_n-q\|^2+3\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2-(1-\alpha_n)\omega_2(b_n)g(\|T^{\lambda_{n+1}}z_n-x_n\|).\end{aligned}\tag{2.13}$$

Thus,

$$(1-\alpha_n)\omega_2(b_n)g(\|T^{\lambda_{n+1}}z_n-x_n\|)\le\|x_n-q\|^2-\|x_{n+1}-q\|^2+3\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2.\tag{2.14}$$

In addition, $(1-\alpha_n)\omega_2(b_n)\ge\alpha(1-\beta)^2$, that is,

$$\alpha(1-\beta)^2g(\|T^{\lambda_{n+1}}z_n-x_n\|)\le\|x_n-q\|^2-\|x_{n+1}-q\|^2+3\frac{\lambda_{n+1}\mu^2}{\tau}\|F(q)\|^2.\tag{2.15}$$

Hence, $\lim_{n\to\infty}g(\|T^{\lambda_{n+1}}z_n-x_n\|)=0$. Since g is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|T^{\lambda_{n+1}}z_n-x_n\|=0$.

Since $\|y_n-x_n\|=(1-b_n)\|T^{\lambda_{n+1}}z_n-x_n\|$, we obtain that $\|y_n-x_n\|\to0$ as $n\to\infty$, and

$$\begin{aligned}\|T^{\lambda_{n+1}}x_n-x_n\|&\le\|T^{\lambda_{n+1}}x_n-T^{\lambda_{n+1}}y_n\|+\|T^{\lambda_{n+1}}y_n-x_n\|\\&\le(1-\lambda_{n+1}\tau)\|x_n-y_n\|+\|T^{\lambda_{n+1}}y_n-x_n\|,\end{aligned}$$

which implies that $\lim_{n\to\infty}\|T^{\lambda_{n+1}}x_n-x_n\|=0$.

(3) It is easy to see that

$$\|Tx_n-x_n\|\le\|Tx_n-T^{\lambda_{n+1}}x_n\|+\|T^{\lambda_{n+1}}x_n-x_n\|\le\|T^{\lambda_{n+1}}x_n-x_n\|+\lambda_{n+1}\mu\|F(Tx_n)\|.$$

Since $\{x_n\}$ is bounded, then $\{Tx_n\}$ and $\{F(Tx_n)\}$ are bounded, too. And $\lambda_n \to 0 (n \to \infty)$, therefore $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. In addition,

$$\|z_n - x_n\| = (1 - a_n)\|T^{\lambda_{n+1}} x_n - x_n\|.$$

By (2), we have $\|z_n - x_n\| \to 0$ as $n \to \infty$. On the other hand,

$$\|Tz_n - z_n\| \le \|Tz_n - Tx_n\| + \|Tx_n - x_n\| + \|x_n - z_n\| \le 2\|z_n - x_n\| + \|Tx_n - x_n\|,$$

so, $\lim_{n\to\infty} \|Tz_n - z_n\| = 0$. By using the same method above, we also can show that $\lim_{n\to\infty} \|Ty_n - y_n\| = 0$. This completes the proof. □

Theorem 2.2. *Let H be a Hilbert space, $T : H \to H$ a nonexpansive mapping with $F(T) \ne \emptyset$, and $F : H \to H$ an η-strongly monotone and k-Lipschitzian mapping. For any given $x_0 \in H$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are defined by (1.3) with the following conditions:*

(i) $\alpha \le a_n, b_n, \alpha_n \le \beta$ *for some* $\alpha, \beta \in (0, 1)$;
(ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$;
(iii) $0 < \mu < 2\eta/k^2$.

Then,

(1) *If $\{x_n\}$ converges strongly to a fixed point of T, then* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$;
(2) *If* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, *then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge strongly to a fixed point of T.*

Proof. (1) Suppose that $\{x_n\}$ converges strongly to a fixed point q of T, then $\lim_{n\to\infty} \|x_n - q\| = 0$. Since $0 \le d(x_n, F(T)) \le \|x_n - q\|$, we know that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

(2) Let $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. For any $p \in F(T)$, we have

$$\|F(p)\| \le \|F(p) - F(x_n)\| + \|F(x_n)\| \le \|x_n - p\| + \|F(x_n)\|.$$

Since $\{F(x_n)\}$ is bounded, there exists a constant $M > 0$ such that $\|F(x_n)\| \le M$. From (2.9),we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|x_n - q\|^2 + 3\frac{\lambda_{n+1}\mu^2}{\tau}\|F(p)\|^2 \\
&\le \|x_n - p\|^2 + 3\frac{\lambda_{n+1}\mu^2}{\tau}(k\|x_n - p\| + \|F(x_n)\|)^2 \\
&\le \|x_n - p\|^2 + 3\frac{\lambda_{n+1}\mu^2}{\tau}(2k^2\|x_n - p\|^2 + 2\|F(x_n)\|^2) \\
&= \left(1 + 6k^2\frac{\lambda_{n+1}\mu^2}{\tau}\right)\|x_n - p\|^2 + 6\frac{\lambda_{n+1}\mu^2}{\tau}\|F(x_n)\|^2. \qquad (2.16)
\end{aligned}$$

From the randomicity of p, we know that

$$[d(x_{n+1}, F(T))]^2 \le \left(1 + 6k^2\frac{\lambda_{n+1}\mu^2}{\tau}\right)[d(x_n, F(T))]^2 + 6\frac{\lambda_{n+1}\mu^2}{\tau}\|F(x_n)\|^2.$$

Since $\{F(x_n)\}$ is bounded, and $\sum_{n=1}^{\infty} \lambda_n < \infty$, we know that $\sum_{n=1}^{\infty} 6k^2 \frac{\lambda_{n+1}\mu^2}{\tau} < \infty$ and $\sum_{n=1}^{\infty} 6\frac{\lambda_{n+1}\mu^2}{\tau}\|F(x_n)\|^2 < \infty$. From Lemma 1.1, it implies that $\lim_{n\to\infty} d(x_n, F(T))$ exists, so $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Setting $M_1 = \max\{2e^{(6k^2\mu^2/\tau)\sum_{i=1}^{\infty}\lambda_i}, 4\frac{\mu^2 M^2}{\tau}e^{(6k^2\mu^2/\tau)\sum_{i=1}^{\infty}\lambda_i}\}$. Then for any $\epsilon > 0$, there exists a positive integer N such that $\sum_{i=n}^{\infty} \lambda_i < \epsilon/4M_1$ and $d(x_n, F(T)) < \sqrt{\epsilon/4M_1}$ as $n \ge N$. Taking $q \in F(T)$, for any $n, m \ge N$, it follows from (2.16) that

$$\begin{aligned}
\frac{\|x_n - x_m\|^2}{2} &\le \|x_n - q\|^2 + \|x_m - q\|^2 \\
&\le \left(1 + 6k^2\frac{\lambda_n\mu^2}{\tau}\right)\|x_{n-1} - q\|^2 + 6\frac{\lambda_n\mu^2}{\tau}\|F(x_{n-1})\|^2 \\
&\quad + \left(1 + 6k^2\frac{\lambda_m\mu^2}{\tau}\right)\|x_{m-1} - q\|^2 + 6\frac{\lambda_m\mu^2}{\tau}\|F(x_{m-1})\|^2 \\
&\le \prod_{i=N+1}^{n}\left(1 + 6k^2\frac{\lambda_i\mu^2}{\tau}\right)\|x_N - q\|^2 + \sum_{i=N+1}^{n-1} 6\frac{\lambda_i\mu^2}{\tau}M^2 \prod_{j=i+1}^{n}\left(1 + 6k^2\frac{\lambda_j\mu^2}{\tau}\right)\|x_N - q\|^2 \\
&\quad + 6\frac{\lambda_n\mu^2}{\tau}M^2 + \prod_{i=N+1}^{m}\left(1 + 6k^2\frac{\lambda_i\mu^2}{\tau}\right)\|x_N - q\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=N+1}^{m-1} 6\frac{\lambda_i\mu^2}{\tau}M^2\prod_{j=i+1}^{m}\left(1+6k^2\frac{\lambda_j\mu^2}{\tau}\right)\|x_N-q\|^2+6\frac{\lambda_m\mu^2}{\tau}M^2\\
&\le 2e^{(6k^2\mu^2/\tau)\sum_{i=1}^{\infty}\lambda_i}\|x_N-q\|^2+4\frac{\mu^2M^2}{\tau}e^{(6k^2\mu^2/\tau)\sum_{i=1}^{\infty}\lambda_i}\sum_{i=N+1}^{\infty}\lambda_i\\
&\le M_1\|x_N-q\|^2+M_1\sum_{i=N+1}^{\infty}\lambda_i.
\end{aligned}
$$

Taking the infimum for all $q \in F(T)$, we have

$$\|x_n-x_m\|^2 \le 2M_1[d(x_N,F(T))]^2+2M_1\sum_{i=N+1}^{\infty}\lambda_i<\epsilon.$$

It follows that $\{x_n\}$ is a Cauchy sequence of the Hilbert space H. Therefore, there exists $p \in H$ such that $\{x_n\}$ converges strongly to p. By Lemma 2.1 we have

$$
\begin{aligned}
\|Tp-p\| &\le \|Tp-Tx_n\|+\|Tx_n-x_n\|+\|x_n-p\|\\
&\le 2\|x_n-p\|+\|Tx_n-x_n\|\to 0 \qquad n\to\infty.
\end{aligned}
$$

Thus, $p \in F(T)$.

From the proof of Lemma 2.1, we have $\|y_n-x_n\|\to 0$, and $\|z_n-x_n\|\to 0$ as $n\to\infty$. So

$$\|y_n-p\|\le\|y_n-x_n\|+\|x_n-p\|\to 0,\quad n\to\infty \tag{2.17}$$

$$\|z_n-p\|\le\|z_n-x_n\|+\|x_n-p\|\to 0,\quad n\to\infty. \tag{2.18}$$

Hence, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ all converge strongly to a fixed point of T. This completes the proof. □

Corollary 2.3. *Under the conditions of Lemma 2.1, if T is completely continuous, then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ all converge strongly to a fixed point of T.*

Proof. It follows from Lemma 2.1 that $\{x_n\}$, $\{Tx_n\}$ are bounded, and $\lim_{n\to\infty}\|Tx_n-x_n\|=0$. Since T is completely continuous, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $\|Tx_{n_k}-q\|\to 0$ as $k\to\infty$. In addition, $\lim_{k\to\infty}\|Tx_{n_k}-x_{n_k}\|=0$. By the continuity of T and Lemma 1.4, we know that $q\in F(T)$, and $\lim_{k\to\infty}\|x_{n_k}-q\|=0$. Since $\lim_{n\to\infty}\|x_n-q\|$ exists by Lemma 2.1, so

$$\lim_{n\to\infty}\|x_n-q\|=\lim_{k\to\infty}\|x_{n_k}-q\|=0.$$

Since $\|y_n-x_n\|\to 0$ and $\|z_n-x_n\|\to 0$ as $n\to\infty$, it follows from (2.17) and (2.18) that $\{y_n\}$, $\{z_n\}$ also converge strongly to a fixed point of T. This completes the proof. □

Corollary 2.4. *Under the conditions of Lemma 2.1, if T is semicompact, then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge strongly to a fixed point of T.*

Proof. By Lemma 2.1, $\{x_n\}$ are bounded, and $\lim_{n\to\infty}\|Tx_n-x_n\|=0$. Since T is semicompact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to $q\in H$. From Lemma 1.4, we have that $q\in F(T)$. In addition, since $\lim_{n\to\infty}\|x_n-q\|$ exists, therefore $\lim_{n\to\infty}\|x_n-q\|=0$, that is, $\{x_n\}$ converges strongly to a fixed point of T. From the proof of Corollary 2.3, we know that $\{y_n\}$, $\{z_n\}$ also converge strongly to a fixed point of T. □

In order to prove the next theorem, we introduce a definition.

Definition 2.5 (*[7]*). A mapping $T:K\to K$ with $F(T)\neq\emptyset$ is said to satisfy condition (A) if there exists a nondecreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(t)>0$ for all $t\in(0,\infty)$ such that $\|x-Tx\|\ge f(d(x,F(T)))$ for all $x\in K$, where $d(x,F(T))=\inf\{\|x-1\|:q\in F(T)\}$.

Theorem 2.6. *Under the conditions of Lemma 2.1, if T satisfies condition (A), then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge strongly to a fixed point of T.*

Proof. Since T satisfies condition (A), then $f(d(x_n,F(T)))\le\|x_n-Tx_n\|$. It follows from Lemma 2.1 that $\lim_{n\to\infty}\|Tx_n-x_n\|=0$, so $\liminf_{n\to\infty}d(x_n,F(T))=0$. By Theorem 2.2, we know that $\{x_n\}$ converge strongly to a fixed point of T. Hence, $\{y_n\}$, $\{z_n\}$ also converge strongly to a fixed point of T. The proof is completed. □

For proving weak convergence for hybrid Noor iteration scheme for nonexpansive mapping in Hilbert spaces, we need the following lemma.

Lemma 2.7. *Let H be a Hilbert space,and $\{x_n\}$ a sequence in H. Let $u,v\in H$ be such that $\lim_{n\to\infty}\|x_n-u\|$ and $\lim_{n\to\infty}\|x_n-v\|$ exists. If $\{x_{n_j}\}$ and $\{x_{n_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u=v$.*

Proof. Suppose that $u \neq v$. Since H is a Hilbert space, it satisfies *Opial's* condition. Thus we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \lim_{j\to\infty} \|x_{n_j} - u\| < \lim_{j\to\infty} \|x_{n_j} - v\| \\ &= \lim_{n\to\infty} \|x_n - v\| = \lim_{k\to\infty} \|x_{n_k} - v\| \\ &< \lim_{k\to\infty} \|x_k - u\| = \lim_{n\to\infty} \|x_n - u\|\end{aligned}$$

which is a contradiction. The proof is completed. $\square$

Theorem 2.8. *Let H be a Hilbert space, $T : H \to H$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $F : H \to H$ an η-strongly monotone and k-Lipschitzian mapping. For any given $x_0 \in H$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are defined as in (1.3) with the following conditions:*

(i) $\alpha \leq a_n, b_n, \alpha_n \leq \beta$ *for some* $\alpha, \beta \in (0, 1)$;
(ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$;
(iii) $0 < \mu < 2\eta/k^2$.

Then, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge weakly to a fixed point of T.

Proof. It follows from Lemma 2.1 that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ and $\{x_n\}$ is bounded, we may assume that $\{x_n\} \to u$ as $n \to \infty$, without loss of generality. By Lemma 1.4, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_j}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 1.4, $u, v \in F(T)$. By (1) of Lemma 2.1, $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 2.7 that $u = v$. Therefore $\{x_n\}$ converges weakly to $u \in F(T)$.

Since $\{x_n\}$ converges weakly to u, for any bounded linear function f, we have $\lim_{n\to\infty} \|f(x_n) - f(u)\| = 0$, thus

$$\begin{aligned}\|f(z_n) - f(u)\| &\leq \|f(z_n) - f(x_n)\| + \|f(x_n) - f(u)\| \\ &\leq \|f\| \|z_n - x_n\| + \|f(x_n) - f(u)\| \to 0 \quad (n \to \infty)\end{aligned}$$

and

$$\begin{aligned}\|f(y_n) - f(u)\| &\leq \|f(y_n) - f(x_n)\| + \|f(x_n) - f(u)\| \\ &\leq \|f\| \|y_n - x_n\| + \|f(x_n) - f(u)\| \to 0 \quad (n \to \infty).\end{aligned}$$

Hence, $\{y_n\}$, $\{z_n\}$ also converge weakly to $u \in F(T)$. This completes the proof. $\square$

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